2. Representations basics (S = finite regular monoid)

- k = field, V = finite dim. vector space / k.

$\text{End}(V)$ = monoid of all linear maps $V \to V$ under composition

An <u>S-action</u> or <u>linear representation of S</u> a (monoid) homom.

$$S \xrightarrow{\varphi} \text{End}(V). \qquad (\Rightarrow 1_S \overset{\varphi}{\mapsto} \text{id} \in \text{End}(V))$$

<u>note</u>: (i) $\text{im}\,\varphi \neq \{0\}$. (ii). If S a group then $\text{im}\,\varphi \subseteq GL(V)$ = gp. invertible linear maps $V \to V$.

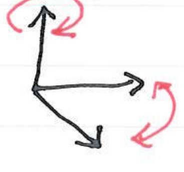
<u>abuses</u>: identify $a \in S$ and $(a)\varphi \in \text{End}(V)$; for $v \in V$ write $v \cdot a$ or va for effect of $(a)\varphi$ on v; say V is an S-representation.

- <u>Eg</u> (mapping representations): V = k-space with basis $\{v_1, \ldots, v_n\}$ and $a \in S_n, I_n$ or T_n. Define

$$v_i \cdot a = v_{ia} \quad (S_n \text{ or } T_n) \qquad \text{or} \qquad v_i \cdot a = \begin{cases} v_{ia}, & i \in \text{dom}\, a \\ 0, & \text{else.} \end{cases} \quad (I_n)$$

and extend linearly.

① S_n (permutation representation)

$n=3$:  $a = (1,2)$ reflection in plane $x_1 - x_2 = 0$

Let $U = k$-span of $v_1 + \cdots + v_n$; then U subspace of V with $US_n \subseteq U$

[*in general*: V an S-rep. and U subspace with $US \subseteq U$; then U an (S-)*subrepresentation* of V.

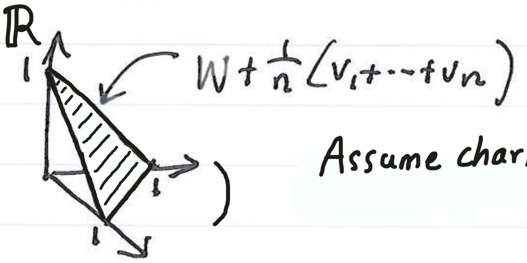
V *irreducible* S-rep. $\overset{\text{def}^n}{\iff}$ the only sub-reps. are $\{0\}$ and V

(*reducible* otherwise).]

Thus S_n-rep. V above reducible $(n>1)$. As $\dim U = 1$ the only subspaces of U are $\{0\}$ or $U \Rightarrow U$ irreducible subrep. of V.

Let W = hyperplane with equation $x_1 + \cdots + x_n = 0$

(Eg: $k = \mathbb{R}$



)

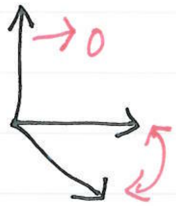
Then W also a subrep. of V

Assume char $k \nmid n$. Then it turns out that W is irreducible.

[*in general*: if U, W subreps. of V with $V = U \oplus W$ as vector spaces, then say V a *direct sum of subreps.*]

Thus $V = U \oplus W$ direct sum irred. subreps. (char $k \nmid n$) Such a V is *completely reducible*.

② I_n (partial permutation rep.)
$(n>1)$

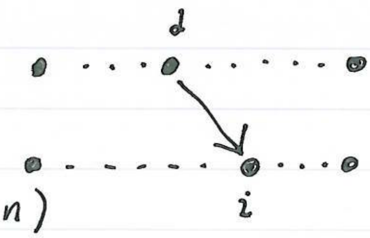


$a = \begin{pmatrix} 1 & 2 & 3 \cdots n \\ 2 & 1 & \text{———} \end{pmatrix}$

$U I_n \not\subseteq U$ and
$W I_n \not\subseteq W$

Indeed, let $V' \subseteq V$ be a subrepresentation. If $v \in V'$ with $v \neq 0$ then $v = \sum \lambda_i v_i$ with some $\lambda_j \neq 0$. If

$a_i =$



$(1 \leq i \leq n)$

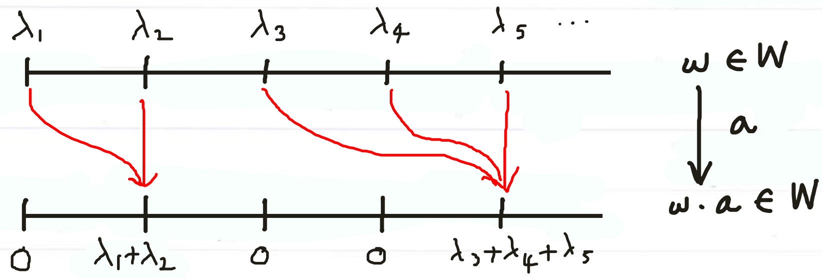
then $v \cdot a_i = \lambda_j v_i$

$\Rightarrow$ each $v_i \in V' \Rightarrow V' = V$.

Thus the partial permutation rep. is irreducible.

③ T_n (mapping rep.)
$(n>1)$

W a subrepresentation: Let $w \in W$ where $w = \sum \lambda_i v_i$ with $\sum \lambda_i = 0$ and $a \in T_n$.



<u>Ex</u>: $T \leq S$ submonoid and V an S-rep $\overset{\text{restrict}}{\Longrightarrow}$ V a T-rep.

Moreover W irreducible: if $W' \subset W$
(char $k \nmid n$)

subrep. then $(S_n \leq T_n)$ have W' a S_n-subrep. of

S_n-rep $W \overset{W \text{ irred.}}{\underset{S_n\text{-rep.}}{\Longrightarrow}} W' = 0$ or W.

On the other hand $UT_n \not\subseteq U$; indeed:

<u>Ex</u>: V has no 1-dim. subreps. $(n>2)$ or exactly one, namely W $(n=2)$.

[A map $V \xrightarrow{\alpha} U$ of S-representations is a linear map that commutes with the S-actions, i.e: $\forall s \in S$

$$\begin{array}{ccc} V & \xrightarrow{(-)s} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ U & \xrightarrow{(-)s} & U \end{array}$$

commutes, α is an <u>isomorphism</u> if bijective.

(more generally: $s \mapsto t$ isom. $S \cong T$ and U a T-representation with $\begin{array}{ccc} V & \xrightarrow{(-)s} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ U & \xrightarrow{(-)t} & U \end{array}$)

<u>Fact</u>: V an S-rep. and $V = \bigoplus_i V_i$ with the V_i irreducible sub-reps. If $W \subset V$ an irreducible sub-rep. then $W \cong V_j$ for some j.]

Back to mapping rep. V of T_n: if $V = \bigoplus_i V_i$ irreducibles then one is $\cong W$, hence $(n-1)$-dimensional $\Rightarrow$ have $V = V_1 \oplus V_2$ with V_1 (say) a 1-dimensional sub-rep.

$(n>2)$ $\overset{\text{no such}}{\underset{\text{exist}}{\Longrightarrow}}$ V cannot be decomposed, i.e: is not completely reducible.

In general, if S a (finite regular) monoid and k a field, then (S,k) <u>semisimple</u> $\overset{\text{def}}{\Longleftrightarrow}$ every S-rep. V (over k) is completely reducible, i.e.: $V = \underbrace{V_1 \oplus \cdots \oplus V_n}_{\text{irred. subreps.}}$

<u>Theorem</u> (Mashke): S a group. Then (S,k) s.s. $\Longleftrightarrow$ char $k \nmid |S|$.

<u>Eg</u>: $S = S_n$ then (S_n, k) s.s. $\Leftrightarrow$ char $k \nmid n!$

<u>Theorem</u>: S an inverse monoid.

Then (S,k) s.s. $\Leftrightarrow$ char $k \nmid |G|$ for $G \leq S$ subgp.

$\Leftrightarrow$ char $k \nmid |H_e|$, any idempotent e.

<u>Eg</u>: $S = I_n$ and $e : \{1,\ldots,m\} \xrightarrow{id} \{1,\ldots,m\} \Rightarrow H_e \cong S_m$ ($m \leq n$)

$\Rightarrow (I_n, k)$ s.s whenever char $k \nmid n!$

<u>Eg</u>: $S = T_n$ and $n > 2 \Rightarrow (T_n, k)$ <u>not</u> s.s. if char $k \nmid n$.